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We wish to consider the integral equation

$$(1) \qquad f(x) = \frac{i}{2} \int_{-1}^{1} H_0^{(1)}(k|x-t|)\,\varphi(t)\,dt\ .$$

Here $H_0^{(1)}$ denotes the zero order Hankel function of the first kind. k is a non-zero constant with $\operatorname{Re} k \geq 0$, $\operatorname{Im} k \geq 0$. Recall that for small r we have

$$(2) \qquad \frac{i}{2} H_0^{(1)}(kr) = \frac{1}{\pi} \log \frac{1}{r} + h(r)$$

where $h(r)$ and $h'(r)$ are finite at $r = 0$. The equation (1) arises in connection with the solution of the reduced wave equation in the plane slit along the x-axis from -1 to +1 [1].

In [1] the following result was proven: Let h denote the class of complex functions φ which are Hölder continuous in a neighborhood of each point of (-1,1) and further satisfy the condition that near $x = 1$

$$|\varphi(x)| \leq \frac{K}{(1+x)^{\alpha}}, \quad 0 \leq \alpha < 1 \quad \text{and near } x = -1, \quad |\varphi(x)| \leq \frac{K}{(1+x)^{\alpha}}\ .$$

Then given $f(x)$ such that f' is Hölder continuous, equation (1) has a unique solution, $\varphi \in h$. In this paper we will consider equation (1) as a mapping from one Hilbert space into another. We will show that if the domain and range spaces are defined appropriately the integral operator in (1) becomes a one to one continuous mapping of one Hilbert space onto another and hence by Banach's open mapping theorem has a continuous inverse. It will be shown that if f is sufficiently smooth, the solutions found here coincide with those found in [1].

Let $p(t) = (1-t^2)^{-\frac{1}{2}}$, $-1 < t < 1$ and $q(t) = (1-t^2)^{\frac{1}{2}} = \frac{1}{p(t)}$, $-1 < t < 1$. We define three spaces:

$$L_2(p) = \left\{ f \;\middle|\; \int_{-1}^{1} |f|^2 (1-t^2)^{-\frac{1}{2}} dt < \infty \right\};$$

$$L_2(q) = \left\{ f \;\middle|\; \int_{-1}^{1} |f|^2 (1-t^2)^{\frac{1}{2}} dt < \infty \right\};$$

$$W_2^1(q) = \left\{ f \;\middle|\; f \text{ is absolutely continuous on } [-1,1] \text{ and } f' \text{ (which exists a.e. with respect to Lebesgue measure) } \in L_2(q) \right\}.$$

If in $L_2(p)$ we define $\|f\|^2_{L_2(p)} = \int_{-1}^{1} |f|^2 (1-t^2)^{-\frac{1}{2}} dt$ and in $L_2(q)$ we define $\|f\|^2_{L_2(q)} = \int_{-1}^{1} |f|^2 (1-t^2)^{\frac{1}{2}} dt$ then these spaces are Hilbert spaces. In $W_2^1(q)$ we define

$$\|f\|^2_{W_2^1(q)} = \|f\|^2_{L_2(q)} + \|f'\|^2_{L_2(q)}.$$

We then have:

Theorem 1. Under the above norm $W_2^1(q)$ is a Hilbert space.

Proof. We first note that $L_2(q) \subset L_1(-1,1)$ (the usual class of functions integrable over $(-1,1)$ with respect to Lebesgue measure) and the injection is continuous. To see this we note

$$\|f\|_1 = \int_{-1}^{1} |f(t)|\,dt = \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} |f(t)| \sqrt{1-t^2}\,dt$$

$$\leq \left\| \frac{1}{\sqrt{1-t^2}} \right\|_{L_2(q)} \|f\|_{L_2(q)} = \sqrt{\pi}\, \|f\|_{L_2(q)}$$

where we have used the Schwarz inequality in $L_2(q)$.

Now suppose $\{f_n\}$ is a Cauchy sequence in $W_2^1(q)$. In particular $\{f_n'\}$ is Cauchy in $L_2(q)$. Thus $\exists\ g \in L_2(q) \ni \| f_n' - g \|_{L_2(q)} \to 0$.

By the above $f_n', g \in L_1(-1,1)$. Thus $f_n(x) = f_n(-1) + \int_{-1}^{x} f_n'(t)\,dt$.

Hence $f_n(-1) - f_m(-1) = f_n(x) - f_m(x) - \int_{-1}^{x} (f_n'(t) - f_m'(t))\,dt$.

Thus $|f_n(-1) - f_m(-1)|^2 \leq 2|f_n(x) - f_m(x)|^2 + 2\|f_n' - f_m'\|_1^2$. Multiply by $\sqrt{1-t^2}$ and integrate from -1 to 1.

$\frac{\pi}{2}|f_n(-1) - f_m(-1)|^2 \leq 2\|f_n - f_m\|_{L_2(q)}^2 + \pi\|f_n' - f_m'\|_1^2$. Thus

$|f_n(-1) - f_m(-1)|^2 \leq \frac{4}{\pi}\|f_n - f_m\|_{L_2(q)}^2 + 2\pi\|f_n' - f_m'\|_{L_2(q)}^2 \to 0$

as $m,n \to \infty$. Thus $f_n(-1) \to C$ as $n \to \infty$. Let

$f(x) = C + \int_{-1}^{x} g(t)\,dt$. f is absolutely continuous and

$f(x) - f_n(x) = C - f_n(-1) + \int_{-1}^{x} (g(t) - f_n'(t))\,dt$

$|f(x) - f_n(x)|^2 \leq 2|C - f_n(-1)|^2 + 2\|g - f_n'\|_1^2$. Thus

$\|f(x) - f_n(x)\|_{L_2(q)}^2 \leq \pi|C - f_n(-1)|^2 + 2\pi\|g - f_n'\|_{L_2(q)}^2 \to 0$

as $n \to \infty$. Thus $\|f_n - f\|_{W_2^1(q)} \to 0$ as $n \to \infty$. ∎

We now consider the operator defined by (1). Let

$$\psi(x) = \frac{i}{2}\int_{-1}^{1} H_0^{(1)}(k|x-t|)\,\varphi(t)\,dt = (L\varphi)(x). \tag{3}$$

As is pointed out in [1] if φ is Hölder continuous we may differentiate under the integral sign and obtain (in view of (2)) :

$$(4) \qquad \psi'(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)\,dt}{x-t} + \int_{-1}^{1}k(t,x)\varphi(t)\,dt$$

where the first term must be taken as a Cauchy Principal Value and in the second term $k(t,x)$ is a continuous kernel.

We now consider (4) as an equation in $L_2(q)$. Let $F: L_2(q) \longrightarrow L_2(p)$ be defined by $(Ff)(t) = \sqrt{1-t^2}\, f(t)$. Then F is an isometry of $L_2(q)$ onto $L_2(p)$. Define an operator T by

$$(5) \qquad Tg = \frac{1}{\pi}\int_{-1}^{1}\frac{g(t)}{x-t}\cdot\frac{1}{\sqrt{1-t^2}}\,dt\,.$$

Then we have the following theorem [2].

Theorem 2. The operator defined by (5) is a continuous mapping from $L_2(p)$ onto $L_2(q)$. Its null space is one dimensional and is spanned by the function $g(x) \equiv 1$. Further the restriction, T_0, of T to the orthogonal complement $H(p)$ of this null space is an isometry of $H(p)$ onto $L_2(q)$ with inverse mapping

$$T_0^{-1}h = \frac{1}{\pi}\int_{-1}^{1}\frac{h(t)}{t-x}\sqrt{1-t^2}\,dt\,.$$

Thus the mapping $\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{x-t}dt$ can be written as $TF\varphi$. We see that it maps $L_2(q)$ continuously onto $L_2(q)$ with a one dimensional null space spanned by $p(t) = (1-t^2)^{-\frac{1}{2}}$. We recall the definition of the index of an operator S from one linear space X to another linear space Y.

Suppose S has a finite dimensional null space $N(S)$, $\dim N(S) = \alpha(S)$, and that the range of S, $R(S)$, has finite codimension.
$\operatorname{codim} R(S) = \dim Y/R(S) = \beta(S)$ (in which case S is said to be a Fredholm operator). The integer $i(S) = \alpha(S) - \beta(S)$ is called the index of the operator S. Thus we have that TF is a Fredholm operator with $\alpha(TF) = 1$, $\beta(TF) = 0$. Thus $i(TF) = 1$. Since $k(t,x)$ is continuous so that

$$\int_{-1}^{1} \int_{-1}^{1} [k(t,x)]^2 \left(\sqrt{1-t^2}\right)^{-1} \sqrt{1-x^2}\, dx\, dt < \infty$$

$\int_{-1}^{1} k(t,x)\, \varphi(t)\, dt$ represents a compact operator, K_0, from $L_2(q)$ into $L_2(q)$. Now the operator TF admits a left regularization [3], i.e. there exists a linear bounded operator Q mapping $L_2(q)$ into $L_2(q)$ such that

$$Q(TF) = I + K$$

where I is the identity in $L_2(q)$ and K is a compact operator (we take $Q = F^{-1} T_0^{-1}$. Then $K = -P_0$ where P_0 is the projection onto the space spanned by $P(t) = \dfrac{1}{\sqrt{1-t^2}}$. We then note:

Theorem 3 [3]. If a bounded operator A admits a left regularization and has finite index and K is any compact operator we have

$$i(A + K) = i(A).$$

Hence we conclude that mapping defined by the right hand side of (4) is a continuous mapping of $L_2(q)$ into $L_2(q)$ with index equal to 1.

We return now to the operator L defined by (3). We have

$\int_{-1}^{1}\int_{-1}^{1} |H_0^{(1)}(k|x-t|)|^2 \left(\sqrt{1-t^2}\right)^{-1}\sqrt{1-x^2}\,dt\,dx < \infty$. Thus L is a continuous (compact) operator from $L_2(q)$ into $L_2(q)$.

Theorem 4. The operator L maps $L_2(q)$ into $W_2^1(q)$.

Proof. Given $\varphi \in L_2(q)$. Let

$$\psi = L\varphi$$

$$\chi = TF\varphi + K_0\varphi .$$

Let $\{\varphi_n\}$ be a sequence of Hölder continuous functions $\ni$ $\|\varphi_n - \varphi\|_{L_2(q)} \longrightarrow 0$. Let $\psi_n = L\varphi_n$.

Then we know that ψ_n is differentiable on $(-1,1)$ and

$$\psi_n' = TF\varphi_n + K_0\varphi_n .$$

By continuity of the mappings L and $TF + K_0$ we see that $\{\psi_n\}$ and $\{\psi_n'\}$ are Cauchy sequences in $L_2(q)$ i.e. $\{\psi_n\}$ is a Cauchy sequence in $W_2^1(q)$. By Theorem 1 $\exists$ a $\psi_0 \in W_2^1(q) \ni \|\psi_n - \psi_0\|_{W_2^1(q)} \longrightarrow 0$. Hence $\|\psi_n - \psi_0\|_{L_2(q)} \longrightarrow 0$ but $\psi_n \longrightarrow \psi$ in $L_2(q)$. Thus $\psi = \psi_0$ a.e. In fact $\psi \equiv \psi_0$ since ψ can easily be shown to be continuous and ψ_0 is absolutely continuous. Also $\chi = \psi_0'$ a.e. Hence the theorem is proven.

Theorem 5. The operator L is a one to one map of $L_2(q)$ onto $W_2^1(q)$.

Proof. Let $f \in W_2^1(q)$ and consider the equation in $L_2(q)$

(6) $$f' = (TF + K_0)\varphi .$$

We know that the index of $(TF + K_0)$ is 1. Thus $\alpha(TF + K_0) \geq 1$. Let $\varphi_0 \in L_2(q)$ satisfy the equation

(7) $$TF\varphi_0 + K_0\varphi_0 = 0 .$$

Recall that $K_0\varphi_0 = \int_{-1}^{1} k(t,x)\varphi(t)\,dt$, $k(t,x) = h'(|t-x|) \sim (t-x)\log|t-x|$.

$k(t,x)$ is Hölder continuous in x uniformly in t (see [4] p. 17). Thus an easy argument shows that if $\varphi_0 \in L_2(q)$, $K_0\varphi_0$ is Hölder continuous. Thus applying the operator $F^{-1}T_0^{-1}$ we see that

$$\varphi_0(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}} \int_{-1}^{1} \frac{(K_0\varphi_0)(t)}{t-x}\sqrt{1-t^2}\,dt + \frac{C}{\sqrt{1-t^2}}$$

but from this we see that $\varphi_0 \in h$. Hence all solutions of (7) in $L_2(q)$ are at the same time in h. Hence applying arguments as in [1] we see that there exists exactly 1 linearly independent solution of (6) in $L_2(q)$, say ϕ_0. Further $L\phi_0 \equiv C_0$ where C_0 is a non zero constant. Thus $\alpha(TF + K_0) = 1$, $\beta(TF + K_0) = 0$, i.e. $TF + K_0$ is onto. Let φ_f be a solution of (6). Then we consider the function $f - L\varphi_f$. This is a function in $W_2^1(q)$ with derivative $f' - (TF + K_0)\varphi_f = 0$ a.e. Thus $f - L\varphi_f = C_f$ where C_f is a definite constant. Thus $\varphi^* = \varphi_f + \frac{C_f}{C_0}\varphi_0$ satisfies $L\varphi^* = f$. The above argument shows that this solution is unique. ∎

Theorem 6. L^{-1} is a continuous mapping from $W_2^1(q)$ onto $L_2(q)$.

Proof. Apply Banach's open mapping theorem.

Finally we note that if f' is Holder continuous and φ is the solution of $L\varphi = f$ we have $(TF + K)\varphi = f'$ and applying the operator $F^{-1}T_0^{-1}$ as is the proof of Theorem 5 we again see that $\varphi \in h$. Hence the solutions found here coincide with those found in [1].

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